

## Game theory

### Problem list 2

1. Find Nash equilibrium points (NE points) in the sets of pure strategies for games described in Problems 1–5 from List 1.
2. Show that in a two player zero-sum game, a strategy pair  $(\bar{x}, \bar{y})$  is an NE point if and only if

$$W_1(x, \bar{y}) \leq W_1(\bar{x}, \bar{y}) \leq W_1(\bar{x}, y) \quad \text{for all } x \in X, y \in Y.$$

3. Check if there exist saddle points for each of the games described in Problems 6–10 from List 1.
4. Prove the Brouwer Fixed Point Theorem for a continuous function  $\Gamma: A \rightarrow A$ , where  $A$  is a convex, bounded and closed subset of the real line. Hint: deduce that  $A = [a, b]$ , and consider the function  $g(x) = \Gamma(x) - x$ .
5. Show that the set of mixed strategies of a player is convex, bounded and closed.
6. Prove the following Lemma:  
If  $X = \{x = (x_1, \dots, x_n), \sum_{i=1}^n x_i = 1, x_i \geq 0\}$ , then for any  $s_1, \dots, s_n$  we have

$$\max_{x \in X} \sum_{i=1}^n x_i s_i = \max\{s_1, \dots, s_n\}.$$

7. For the following matrix games, find all NE points:

$$M = \begin{pmatrix} (0, 2) & (1, 0) \\ (1, 0) & (0, 2) \end{pmatrix}, \quad N = \begin{pmatrix} (1, 1) & (-1, 2) \\ (2, -1) & (0, 0) \end{pmatrix}, \quad P = \begin{pmatrix} (0, -1) & (1, 1) \\ (2, 2) & (0, 0) \end{pmatrix},$$

$$Q = \begin{pmatrix} (4, 2) & (1, 7) & (6, 4) \\ (4, 4) & (4, 5) & (2, 3) \\ (5, 5) & (2, 2) & (5, 2) \end{pmatrix}.$$

What payoffs can the players guarantee for themselves by using these NE points?

8. Using the support version of the Nash Theorem, find all NE points for the game:

$$R = \begin{pmatrix} (7, 2) & (2, 7) & (3, 6) \\ (1, 5) & (1, 6) & (2, 4) \\ (2, 7) & (7, 2) & (4, 5) \end{pmatrix}$$

9. Check if the games below have NE points with supports of the form

(a)  $\{1, 3\} \times \{1, 2\}$ ;

(b)  $\{1, 2, 3\} \times \{1, 2, 3\}$ ;

$$V = \begin{pmatrix} (0, -1) & (0, 2) & (2, 3) \\ (0, 0) & (2, 1) & (1, -1) \\ (2, 2) & (1, 4) & (1, -1) \end{pmatrix}, \quad W = \begin{pmatrix} (3, 3) & (0, 2) & (0, 2) \\ (2, 0) & (2, 2) & (2, 0) \\ (2, 0) & (0, 2) & (3, 3) \end{pmatrix}.$$

Find also all NE points for the sets of pure strategies in these games.

10. Allowing the possibility of collaboration between players, check if the NE points for the sets of pure strategies found in the previous problems are Pareto-optimal.
11. Show that in a zero-sum game, all strategies are Pareto-optimal.