

## Game theory

### Problem list 4

**All problems on this list concern two player zero-sum games.**

1. Show that adding a constant to all of the entries in the payoff matrix does not change the optimal strategy sets for either player. How does the game value change?  
And what if we multiply the entries of the payoff matrix by a non-zero constant?
2. A family of matrices depending on a single real parameter  $s$  is given:

$$M(s) = \begin{pmatrix} 0 & 2 & 4 & -5 & 3 \\ 7 & 1+s & -2 & 0 & 4 \\ 5 & 3 & 4+s & 5 & 4 \\ 1 & -3 & 0 & 6 & -6 \end{pmatrix}.$$

Find the set  $S$  of all real numbers  $s$  for which the matrix  $M(s)$  has a saddle point.

3. Prove that the game value is not lower (resp. not higher) than the lower value (resp. the upper value) determined for the sets of pure strategies.
4. Check if the vectors

$$x = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \quad \text{and} \quad y = \left(\frac{1}{4}, \frac{5}{8}, \frac{1}{8}\right)$$

are optimal strategies for the game with payoff matrix

$$W = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 3 & 1 \\ 2 & 2 & 4 \end{pmatrix}.$$

If they are then give the game value, and if not then give lower and upper bounds for it.

5. Check if the vectors

$$x = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \quad \text{and} \quad y = \left(\frac{1}{4}, \frac{5}{8}, \frac{1}{8}\right)$$

are optimal strategies for the game with payoff matrix

$$V = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

If they are then give the game value, and if not then give lower and upper bounds for it.

6. Find a solution to each of the games given by the following payoff matrices:

$$A = \begin{pmatrix} -2 & 2 & 6 & 5 & 0 \\ 8 & 4 & 0 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & 2a & \frac{1}{2} & 2a & \frac{1}{4} & \dots \\ a & 1 & 2a & \frac{1}{3} & 2a & \dots \end{pmatrix} \text{ for } a > 0,$$

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 3 & -1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 4 & 4 \\ 5 & 3 & 5 \\ 6 & 6 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -1 & 3 \\ -3 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix},$$

$$F = \begin{pmatrix} 1 & 6 & 5 \\ 8 & 4 & 0 \\ 3 & 2 & 7 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -2 & 14 \\ 2 & 10 \\ 8 & 0 \\ 10 & 6 \end{pmatrix},$$

$$I = \begin{pmatrix} 5 & 0 & 4 & -1 \\ 9 & -1 & 4 & -2 \\ 7 & 6 & 6 & 5 \\ 5 & 0 & 5 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 3 & 4 & 4 & 4 \\ 4 & 3 & 4 & 4 \\ 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 8 & 1 & 6 & 0 \\ 8 & 0 & 8 & 2 \\ 4 & 5 & 1 & 6 \\ -1 & 6 & -3 & 6 \\ 7 & 2 & 6 & 2 \end{pmatrix}.$$

7. Given that  $x = (\frac{5}{8}, 0, 0, \frac{1}{4}, 0, \frac{1}{8})$  is an optimal strategy for the first player in the game given by the matrix

$$M = \begin{pmatrix} a & b \\ 2 & 2 \\ -4 & 6 \\ 2 & 3 \\ 4 & 0 \\ 1 & 4 \end{pmatrix},$$

find the values of the parameters  $a$  and  $b$ , an optimal strategy for the second player, and the game value.

8. Find all solutions to the game given by the matrix

$$N = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

9. Find all the extremal optimal strategies and the game value for the game given by the matrix

$$W = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 3 & -1 & -3 \end{pmatrix}.$$