

Game theory

Nash Theorems

A few versions of the Nash Theorem (also called the Minimax Theorem) have been presented at the lectures, as below. The first one concerns two player games, the second one multiplayer games, and the third one two player zero-sum games.

We call a game *finite* if the set of pure strategies for each of the players is finite. In this course, almost all the games considered will be finite.

Nash Theorem (regular version). *Every finite two-player game, with sets of mixed strategies X and Y for player I and player II (respectively), contains at least one Nash equilibrium point: that is, a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that $W_1(x, \bar{y}) \leq W_1(\bar{x}, \bar{y})$ for all $x \in X$ and $W_2(\bar{x}, y) \leq W_2(\bar{x}, \bar{y})$ for all $y \in Y$.*

For a player with the set of pure strategies $\{\tilde{x}_1, \dots, \tilde{x}_m\}$, the set of mixed strategies has the form

$$X = \left\{ (x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^m x_i = 1 \right\},$$

where the pure strategy $\tilde{x}_i$ is identified with the tuple $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i -th place. In this case, we define the *support* of a strategy $x = (x_1, \dots, x_m) \in X$ to be

$$\text{supp}(x) = \{i \in \{1, \dots, m\} \mid x_i > 0\}.$$

Nash Theorem (support version). *In a finite n player game with the set of mixed strategies X_i for the i -th player, a collection of strategies $(\bar{x}_1, \dots, \bar{x}_n) \in X_1 \times \dots \times X_n$ is a Nash equilibrium point if and only if for each $i \in \{1, \dots, n\}$ there exists a constant $w_i \in \mathbb{R}$ such that:*

(i) $w_i = W_i(\bar{x}_1, \dots, \bar{x}_{i-1}, \tilde{x}_k, \bar{x}_{i+1}, \dots, \bar{x}_n)$ whenever $k \in \text{supp}(\bar{x}_i)$;

(ii) $w_i \geq W_i(\bar{x}_1, \dots, \bar{x}_{i-1}, \tilde{x}_\ell, \bar{x}_{i+1}, \dots, \bar{x}_n)$ whenever $\ell \notin \text{supp}(\bar{x}_i)$.

Now consider a two player zero-sum game, with the sets of (mixed) strategies of the players denoted by X and Y , respectively. For such a game we may define its *lower value* $\underline{v}$ and its *upper value* $\bar{v}$ as follows:

$$\begin{aligned} \underline{v} &= \max_{x \in X} \min_{y \in Y} W(x, y); \\ \bar{v} &= \min_{y \in Y} \max_{x \in X} W(x, y); \end{aligned}$$

where W denotes the payoffs for the first player. It is not hard to show that $\underline{v} \leq \bar{v}$.

Nash Theorem (minimax version). *For a finite two player zero-sum game with the sets of (mixed) strategies of the players X and Y , respectively, a strategy pair $(\bar{x}, \bar{y}) \in X \times Y$ is optimal (i.e. a Nash equilibrium point) if and only if*

$$\min_{y \in Y} W(\bar{x}, y) = \underline{v} = \bar{v} = \max_{x \in X} W(x, \bar{y}).$$

We call $\underline{v} = \bar{v}$ the *value of the game* and denote it by v .