ON A CENTRAL LIMIT THEOREM FOR SHRUNKEN WEAKLY DEPENDENT RANDOM VARIABLES

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Abstract. A central limit theorem is proved for some strictly stationary sequences of random variables that satisfy certain mixing conditions and are subjected to the "shrinking operators" $U_r(x) := [\max\{|x|-r, 0\}] \cdot x/|x|$, $r \geq 0$. For independent, identically distributed random variables, this result was proved earlier by Housworth and Shao.

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1. Introduction

In the probability theory and mathematical statistics, many classical limit laws deal with the following sequences

$$A_n(X_1 + X_2 + ... + X_n) + b_n = A_nX_1 + ... + A_nX_n + b_n, \tag{1.1}$$

where (X_n) are stochastically independent random variables or random vectors, (A_n) are real numbers or linear operators and (b_n) are some deterministic shifts; cf. Loève [1977], Feller [1971], Jurek and Mason [1993], and Meerschaert and Scheffler [2001]. Thus one could ask for limits in (1.1) with some non-linear deformation instead of the linear operators A_n.

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In the middle of the 1970's, there were introduced the non-linear shrinking operations U_r, $r \geq 0$, on a Hilbert space H, (in short: *s-operations*), as follows:

$$U_r(0) := 0 \text{ and } U_r(x) := [\max\{||x|| - r, 0\}]\frac{x}{||x||} \quad \text{for } x \neq 0.$$

For these operations, the limit laws of sequences

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \ldots + U_{r_n}(X_n) + b_n, \tag{1.2}$$

were completely characterized for infinitesimal triangular arrays; cf. Jurek [1977] [1981]. The limits in (1.2) were called *s-selfdecomposable and s-stable* for only independent X_n's or independent and identically distributed random variables, respectively. Cf. also Jurek [1977] [1979] [1984] [1985].

In Jurek [1981], on page 5 it was said: *The s-operations have some technical justification. For instance, when we receive a signal X then if X is "small", i.e., X is not greater than r, we get zero, and in the remaining cases we get only the excess, i.e.* $X - r$.

Nowadays, in mathematical finance, note that for positive real-valued variables X, $U_r(X) \equiv (X - r)^+$ (positive part) coincides with the pay-off function of European call options; cf. Föllmer and Schied [2011]. Because of these potential applications, Marc Yor on a couple of occasions asked one of the authors (Z.J.J.) about possible results on limits in (1.2) without the assumption of independence. This note will provide an answer involving a Gaussian limit in (1.2) under certain mixing assumptions.

It is interesting to note that the class $\mathcal{U}$ of all possible limit laws in (1.2) coincides with the totality of probability distributions of the following random integrals:

$$\int_{(0,1]} t\, dY(t), \quad \text{where } Y \text{ is an arbitrary Lévy process,} \tag{1.3}$$

cf. Jurek [1984]. From (1.3), taking Y to be Brownian motion, we see that the Gaussian probability measure can appear as a limit in (1.2); also cf. Jurek [1981], Lemma 5.2.

For real-valued random variables, Housworth and Shao [2000] proved a central limit theorem for these "shrinking operators" and stochastically independent, identically distributed random variables. In this note we prove a CLT (Theorem 2.1 below) as in (1.2) for sequences (X_n) that satisfy some strong mixing and stationarity conditions.

2. Basic notions, notations and results

All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. For any given σ-field $\mathcal{A} \subset \mathcal{F}$, $L_2(\mathcal{A})$ denotes the family of all square-integrable, $\mathcal{A}$-measurable random variables. By $E[X]$ we denote the expected value (if it is defined) of a given random variable X; and by $\|X\|_2$ we denote the $\mathcal{L}^2$-norm of a given $X \in L_2(\mathcal{F})$.

For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, define the following two measures of dependence:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}; \tag{2.1}$$

$$\rho(\mathcal{A}, \mathcal{B}) := \sup\{|Corr(f, g)| : f \in L_2(\mathcal{A}), g \in L_2(\mathcal{B})\} \tag{2.2}$$

where $Corr$ stands for the correlation.

The quantity $\rho(\mathcal{A}, \mathcal{B})$ is called the "maximal correlation coefficient" of $\mathcal{A}$ and $\mathcal{B}$. For every pair of σ-fields $\mathcal{A}$ and $\mathcal{B}$, one has (see e.g. Bradley [2007, v1, Proposition 3.11(a)]) the well known inequalities

$$0 \le 4\alpha(\mathcal{A}, \mathcal{B}) \le \rho(\mathcal{A}, \mathcal{B}) \le 1. \tag{2.3}$$

Now suppose that $\mathbf{X} := (X_k, k \in \mathbf{Z})$ is a *strictly stationary* sequence of (real-valued) random variables. (That is, for any integers j and ℓ and any nonnegative integer m, the random vectors $(X_j, X_{j+1}, \dots, X_{j+m})$ and $(X_\ell, X_{\ell+1}, \dots, X_{\ell+m})$ have the same distribution.) For each positive integer n, define the following three dependence coefficients:

$$\alpha(n) = \alpha(\mathbf{X}, n) := \alpha(\sigma(X_k, k \le 0), \sigma(X_k, k \ge n)); \tag{2.4}$$

$$\rho(n) = \rho(\mathbf{X}, n) := \rho(\sigma(X_k, k \le 0), \sigma(X_k, k \ge n)); \tag{2.5}$$

$$\rho^*(n) = \rho^*(\mathbf{X}, n) := \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)), \tag{2.6}$$

where in (2.6) the supremum is taken over all pairs of nonempty, disjoint sets $S, T \subset \mathbb{Z}$ such that $\text{dist}(S, T) := \min_{s \in S, t \in T} |s - t| \ge n$. In (2.4), (2.5), and (2.6) and below, the notation $\sigma(\dots)$ refers to the σ-field of events generated by $(\dots)$. In (2.6) the two sets S and T can be "interlaced", with each one having elements between ones in the other set. For each $n \in \mathbb{N}$, by (2.3)–(2.6),

$$0 \le 4\alpha(n) \le \rho(n) \le \rho^*(n) \le 1. \tag{2.7}$$

The (strictly stationary) sequence $\mathbf{X} = (X_k, k \in \mathbb{Z})$ is said to be "strongly mixing" (or "α-mixing") if $\alpha(n) \to 0$ as $n \to \infty$, "ρ-mixing" if $\rho(n) \to 0$ as

$n \to \infty$, and "ρ^*-mixing" if $\rho^*(n) \to 0$ as $n \to \infty$. By (2.7), ρ-mixing implies strong mixing, and ρ^*-mixing implies ρ-mixing.

The strong mixing condition was introduced by Rosenblatt [1956], the ρ-mixing condition was introduced by Kolmogorov and Rozanov [1960], and the ρ^*-mixing condition was apparently first studied by Stein [1972]. (The maximal correlation coefficient $\rho(.,.)$ in (2.2) was first studied by Hirschfeld [1935] in a statistical context not particularly connected with "stochastic processes".)

Recall that for each nonnegative real number r, the "shrinking operator" $U_r : \mathbf{R} \to \mathbf{R}$ is defined as follows: $U_r(0) := 0$, and for $x \neq 0$

$$U_r(x) \ := \ [\max\{|x| - r, 0\}]\frac{x}{|x|} \ = \ \begin{cases} x - r & \text{if } x > r \\ 0 & \text{if } -r \leq x \leq r \\ x + r & \text{if } x < -r. \end{cases} \tag{2.8}$$

Now we can state the main result of this paper in terms of the all above notions:

Theorem 2.1. *Suppose $\mathbf{X} = (X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables. For each $r \geq 0$, define the quantity*

$$G(r) := \int_0^\infty t \cdot P(|X_0| > t + r)dt. \tag{2.9}$$

Suppose that

$$\forall r \geq 0, \quad 0 < G(r) < \infty \tag{2.10}$$

and that

$$\forall \varepsilon > 0, \quad \lim_{r \to \infty} \frac{G(r + \varepsilon)}{G(r)} = 0. \tag{2.11}$$

Then for each $r \geq 0$, $E[|U_r(X_0)|^2] < \infty$.

Suppose also that at least one of the following two conditions holds:
(i) $\rho(1) < 1$ and $\sum_{n=1}^\infty \rho(2^n) < \infty$; or
(ii) $\rho^*(1) < 1$, and $\alpha(n) \to 0$ as $n \to \infty$.

Then there exists a sequence $(r(n), n \in \mathbb{N})$ of positive numbers satisfying

$$r(n) \to \infty \ \text{ as } \ n \to \infty, \tag{2.12}$$

such that as $n \to \infty$,

$$[U_{r(n)}(X_1) + U_{r(n)}(X_2) + \ldots + U_{r(n)}(X_n)] - \ n \cdot m_{r(n)} \ \Rightarrow \ N(0,1), \tag{2.13}$$

where for each $r \geq 0$, $m_r := E[U_r(X_0)]$.

A few comments on this theorem are in order.

Using Fubini's Theorem (see Remark 3.2(b) in Section 3), one can express conditions (2.10) and (2.11) in terms of second moments as follows

$$\forall r \geq 0, \quad 0 < 2G(r) = \mathbb{E}[|U_r(X_0)|^2] < \infty, \tag{2.14}$$

and

$$\forall \varepsilon > 0, \quad \lim_{r \to \infty} \frac{E[|U_{r+\varepsilon}(X_0)|^2]}{E[|U_r(X_0)|^2]} = 0. \tag{2.15}$$

The proof of Theorem 2.1 will be given in Section 4, after some elementary but important facts concerning the "shrinking operators" $U_r(\,.\,)$ are given in Section 3.

In the case where the random variables X_k are independent and identically distributed, Theorem 2.1 is due to Housworth and Shao [2000, Theorem 1]. (Recall also the earlier related work of Jurek [1982] in connection with (1.3).) In their result, Housworth and Shao [2000, Theorem 1] also gave (implicitly) some concrete extra information on the sequence of "shrinking parameters" $r(n),\ n \in \mathbb{N}$. In our context, because of the dependence between the random variables, the same information cannot be given. Some partial information on the $r(n)$'s can be seen implicitly from the proof (in Section 4) of Theorem 2.1.

In their proof of their version of Theorem 2.1 for independent, identically distributed random variables, Housworth and Shao [2000, Theorem 1] used (2.10) and (2.11) to in essence establish a Lindeberg condition — indirectly, in the context of a "codified central limit theorem" for independent random variables in the book of Petrov [1975]. In our different context, involving dependent random variables, we shall adapt their arguments to verify a Lindeberg condition directly, and then use Lindeberg CLT's in the literature under the dependence conditions (i) and (ii) in Theorem 2.1 respectively to complete the proof of that theorem.

In Theorem 2.1, neither weak dependence condition (i) or (ii) implies the other; see e.g. Bradley [2007, v3, Theorem 26.8(II)]. In central limit theorems for strictly stationary ρ-mixing random sequences with finite second moments, the key role of the "logarithmic mixing rate" condition $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ in condition (i) was established by Ibragimov [1975]; and the sharpness of that condition in that context was shown by counterexamples (with finite second moments and barely slower mixing rates) such as in Bradley [2007, v3, Chapter 34]. In the case of strictly stationary ρ-mixing sequences with (in a certain standard technical sense) "barely infinite second moments" such that both parts of condition (i) hold, a central limit theorem was proved by

Bradley [1988] and extended to a weak invariance principle by Shao [1993]. In other central limit theory under mixing conditions with just finite second moments, a prominent role of the weak dependence condition (ii) (in Theorem 2.1) was established in papers such as Peligrad [1996] [1998] and Utev and Peligrad [2003].

The alternative mixing assumptions (i) and (ii) in Theorem 2.1 provide key bounds on variances of the partial sums $\sum_{k=1}^{n} U_{r(n)}(X_k)$ in the left side of (2.13). (Those bounds are based on Lemmas 6.1 and 6.2 in the Appendix.) An example described in the next theorem will show what can "go wrong" in Theorem 2.1 under other, seemingly nice, mixing conditions that fail to impose such bounds on the variances of those partial sums. It is a "cancellation" example of a type that has long been well known in central limit theory under dependence assumptions. (For an old, classic, very simple "cancellation" example, see e.g. Bradley [2007, v1, Example 1.18].)

Theorem 2.2. *Suppose $0 < \lambda < 1$. Then there exists a strictly stationary sequence $\mathbf{X} := (X_k, k \in \mathbb{Z})$ of random variables that satisfies (2.10) and (2.11) and also has the following three properties:*
(a) $\alpha(\mathbf{X}, 1) \le \lambda$.
(b) For some $c > 0$, $\rho^*(\mathbf{X}, n) = O(e^{-cn})$ as $n \to \infty$.
(c) For every $r > 0$ and every $n \in \mathbb{N}$, $P(\sum_{k=1}^{n} U_r(X_k) = 0) \ge 1 - \lambda$.

By (2.7), analogs of property (b) automatically also hold for the dependence coefficients $\alpha(\mathbf{X}, n)$ and $\rho(\mathbf{X}, n)$. Of course property (c) prevents the random sums in (2.13) from converging to a (non-degenerate) normal law under any choice of the $r(n)$'s and any kind of normalization. Properties (a) and (c) take their most "extreme" forms when the parameter λ is very small. From (2.7) and Theorem 2.1 itself, one can see immediately that in the example described in Theorem 2.2, $\rho(1) = \rho^*(1) = 1$ must hold.

Theorem 2.2 will be proved with a construction in Section 5. (Also, in Remark 5.1 at the end of that section, a direct brief explanation of the equality $\rho(1) = \rho^*(1) = 1$ will be given for that particular construction.)

Remark 2.3. Under its dependence condition (ii), Theorem 2.1 can be extended to a weak invariance principle (we shall not formulate that here) via the combining of our proof (in Section 4) with an application of the main result of Utev and Peligrad [2003]. Such an extension of Theorem 2.1 under its dependence condition (i) to a weak invariance principle, if that holds, would apparently require more effort, apparently involving intricate tightness arguments similar to those in Shao [1989] [1993].

3. Background remarks

For ease of reference, some elementary properties of the shrinking operators and corresponding expected values are collected in the following two remarks. (The proofs are simple and are mostly left to the reader.)

Remark 3.1. For any $x \in \mathbb{R}$ and any $r, s \in [0, \infty)$ one has the following:
(a) $U_r(U_s(x)) = U_{r+s}(x)$, (the one-parameter semigroup property).
(b) $U_0(x) = x$, $U_r(-x) = -U_r(x)$, $|U_r(x)| \downarrow 0$ as $r \uparrow \infty$.
(c) $|U_r(x)| = \max\{|x| - r, 0\}$, $|U_r(x)| \le |x|$.
(d) $|U_s(x) - U_r(x)| \le |s - r|$; in particular, $|x - U_r(x)| \le r$.
(e) $|U_r(x)| \le |s - r| + |U_s(x)|$ (with equality if $|x| \ge s \ge r$).
(f) If $t > 0$, then $|U_r(x)| \ge t$ if and only if $|x| \ge r + t$.
(g) If $\varepsilon > 0$ and $|x| \ge r + \varepsilon$, then $|U_r(x)| \le 2|U_{r+\varepsilon/2}(x)|$. (That holds because $|U_r(x)| \ge \varepsilon$ and hence $|U_r(x)| - |U_{r+\varepsilon/2}(x)| = \varepsilon/2 \le (1/2)|U_r(x)|$.)

Remark 3.2. For a given random variable Y and values $r \in [0, \infty)$, one has the following:
(a) $E[|U_r(Y)|^2] \le E[Y^2]$ (possibly infinity) by Remark 3.1(c). If $E[|U_r(Y)|^2] < \infty$ for some $r \ge 0$, then $E[Y^2] < \infty$ by Remark 3.1(d), and $E[|U_r(Y)|^2] < \infty$ for all $r \ge 0$.
(b) By a well known application of Fubini's Theorem, followed by Remark 3.1(f),

$$E[|U_r(Y)|^2] \;=\; 2\int_0^\infty tP(|U_r(Y)| > t)dt \;=\; 2\int_0^\infty tP(|Y| > t + r)dt. \tag{3.1}$$

(c) If $E[Y^2] < \infty$, then (for example by Remark 3.1(b)(c) and dominated convergence), $E[|U_r(Y)|^2] \to 0$ as $r \to \infty$.

4. Proof of Theorem 2.1

Assume the hypotheses (and notations) in the first paragraph of the statement of Theorem 2.1. As was noted in Section 2, eqs. (2.14) and (2.15) hold by (and are in fact equivalent to) (2.10) and (2.11), by (3.1). Henceforth (2.14) and (2.15) will be used freely. In particular, from (2.14), $E[|U_r(X_0)|^2] < \infty$ for every $r \in [0, \infty)$. As in the statement of Theorem 2.1, for each $r \in [0, \infty)$, define the real number

$$m_r := E[U_r(X_0)] = E[U_r(X_0)I(|X_0| > r)] \tag{4.1}$$

(where the latter equality holds by (2.8)). Now assume that (at least) one of the weak dependence conditions (i) or (ii) in (the third paragraph) of Theorem 2.1 holds. Our remaining task is to prove the last sentence (last paragraph) of Theorem 2.1.

To simplify that argument, we shall convert to an appropriate "mean 0" context. For each $r \in [0, \infty)$, define the random sequence $\mathbf{Y}^{(r)} := (Y_{k,r}, k \in \mathbb{Z})$ as follows: For each $k \in \mathbb{Z}$,

$$Y_{k,r} := U_r(X_k) - m_r. \tag{4.2}$$

For each $r \in [0, \infty)$, this sequence $\mathbf{Y}^{(r)}$ is strictly stationary, by the assumption of the strict stationarity of the sequence $\mathbf{X}$ in Theorem 2.1. Further, for each $r \geq 0$ and each $n \geq 1$, $\alpha(\mathbf{Y}^{(r)}, n) \leq \alpha(\mathbf{X}, n)$ and $\rho(\mathbf{Y}^{(r)}, n) \leq \rho(\mathbf{X}, n)$ and also $\rho^*(\mathbf{Y}^{(r)}, n) \leq \rho^*(\mathbf{X}, n)$ (since for each k and r, $Y_{k,r}$ is a Borel function of X_k). Also, for each $r \geq 0$ and each $k \in \mathbb{Z}$,

$$E[Y_{k,r}] = E[Y_{0,r}] = 0 \quad \text{and} \quad E[Y_{k,r}^2] = E[Y_{0,r}^2] = \mathrm{Var}\,[U_r(X_0)] > 0. \tag{4.3}$$

Here the last inequality (>0) must hold because otherwise one would have $U_r(X_0) = m_r$ a.s. by (4.1), hence $P(|X_0| > |m_r| + r) = 0$, and thus $G(|m_r| + r) = 0$ (see (2.9)), which contradicts (2.10).

To complete the proof of Theorem 2.1, our task is to prove that there exists a sequence $(r(n), n \in \mathbb{N})$ of positive numbers satisfying (2.12) such that

$$\Big[\sum_{k=1}^{n} U_{r(n)}(X_k)\Big] - n \cdot m_{r(n)} \;=\; \sum_{k=1}^{n} Y_{k,r(n)} \;\Rightarrow\; N(0,1) \quad \text{as } n \to \infty. \tag{4.4}$$

Here the equality holds by (4.1) and (4.2). What remains is to prove the convergence to the standard normal law. The proof will involve the random variables $Y_{k,r}$ and will be divided into four "steps": Some preliminary work will be done in Lemma 4.1 and Step 4.2, the sequence $(r(n))$ will be constructed in Lemma 4.3, and then the convergence to the standard normal law in (4.4) will be verified in Step 4.4, thereby completing the proof of Theorem 2.1.

Lemma 4.1. *One has that $m_r^2 = o(E[(U_r(X_0))^2])$ as $r \to \infty$, and (hence)*

$$\frac{E[Y_{0,r}^2]}{2G(r)} = \frac{E[Y_{0,r}^2]}{E[(U_r(X_0))^2]} = \frac{Var[U_r(X_0)]}{E[(U_r(X_0))^2]} \to 1 \ \text{ as } r \to \infty; \tag{4.5}$$

and also one has that

$$\forall \varepsilon > 0, \quad E[Y_{0,r}^2 I(|Y_{0,r}| \geq \varepsilon)] = o(G(r)) \ \text{ as } \ r \to \infty. \tag{4.6}$$

This lemma and its proof given here are a slightly modified version, convenient for our context, of calculations of Housworth and Shao [2000, pp. 263-264].

Proof. To prove (4.5) (and the line preceding it), first observe that by (4.1), for any given $r \geq 0$,

$$\begin{aligned} |m_r| &\leq E[\,|U_r(X_0)| \cdot I(|X_0| > r)] \\ &\leq \big(E[|U_r(X_0)|^2]\big)^{1/2} \cdot \big(E[(I(|X_0| > r))^2]\big)^{1/2} \\ &= \big(E[|U_r(X_0)|^2]\big)^{1/2} \cdot \big(P(|X_0| > r)\big)^{1/2}. \end{aligned}$$

Since $P(|X_0| > r) \to 0$ as $r \to \infty$, one has that $m_r^2 = o(E[(U_r(X_0))^2])$ as $r \to \infty$. And now (4.5) follows from (4.1), (4.2), (4.3), and (2.14).

Now let us prove (4.6). For every $r \geq 0$ and every $\varepsilon > 0$, by Remark 3.1(f)(g) and (2.14), one has that

$$\begin{aligned} E\big(|U_r(X_0)|^2 I(|U_r(X_0)| \geq \varepsilon)\big) &= E\big(|U_r(X_0)|^2 I(|X_0| \geq r + \varepsilon)\big) \\ &\leq E\big[|2U_{r+\varepsilon/2}(X_0)|^2 I(|X_0| \geq r + \varepsilon)\big] \\ &\leq 4E[|U_{r+\varepsilon/2}(X_0)|^2] \;=\; 8G(r + \varepsilon/2). \end{aligned}$$

Hence by (2.11),

$$\forall \varepsilon > 0, \quad E\big([U_r(X_0)]^2 I(|U_r(X_0)| \geq \varepsilon)\big) = o(G(r)) \quad \text{as} \quad r \to \infty. \tag{4.7}$$

Of course from Remark 3.2(c) and (4.1) (or implicitly from above), $m_r \to 0$ as $r \to \infty$. If $\varepsilon > 0$, and $r \geq 0$ is such that $|m_r| \leq \varepsilon/2$, then by (4.2), $\{|Y_{0,r}| \geq \varepsilon\} \subset \{|Y_{0,r}| \leq 2|U_r(X_0)|\}$, and hence

$$\begin{aligned} E[Y_{0,r}^2 I(|Y_{0,r}| \geq \varepsilon)] &\leq E[(2|U_r(X_0)|)^2 I(|Y_{0,r}| \geq \varepsilon)] \\ &\leq E[(2|U_r(X_0)|)^2 I(2|U_r(X_0)| \geq \varepsilon)] \\ &= 4E[|U_r(X_0)|^2 I(|U_r(X_0)| \geq \varepsilon/2)]. \end{aligned}$$

Hence by (4.7), eq. (4.6) holds. That completes the proof of Lemma 4.1.

Step 4.2. Referring to the Appendix in Section 6, applying Lemma 6.1 or Lemma 6.2 (depending on which of the dependence assumptions (i) or (ii) in Theorem 2.1 is assumed), let C be a positive constant such that

$$\forall r \geq 0, \ \forall n \in \mathbb{N}, \quad (1/C)nE[Y_{0,r}^2] \leq E\Big[\Big(\sum_{k=1}^{n} Y_{k,r}\Big)^2\Big] \leq CnE[Y_{0,r}^2]. \tag{4.8}$$

Referring to (4.3), let N_0 be a positive integer such that

$$(1/C)N_0\, E[Y_{0,0}^2] > 1. \tag{4.9}$$

(Note that by (4.2) and (2.8), $Y_{0,0} = X_0 - m_0 = X_0 - E[X_0]$.)

Lemma 4.3. *For each integer $n \geq N_0$, there exists a positive number $r(n)$ such that*

$$\Big\|\sum_{k=1}^{n} Y_{k,r(n)}\Big\|_2 = 1. \tag{4.10}$$

Further, if $(r(n), n \geq N_0)$ is a sequence of positive numbers such that (4.10) holds for all $n \geq N_0$, then

$$r(n) \to \infty \quad \text{as } n \to \infty, \tag{4.11}$$

and

$$\forall n \geq N_0, \quad 1/(Cn) \leq E[Y_{0,r(n)}^2] \leq C/n. \tag{4.12}$$

Proof. We shall first prove the first sentence (with eq. (4.10)) of Lemma 4.3. Suppose $0 \leq r < s$. Then for every $k \in \mathbb{Z}$, one has that $|U_r(X_k) - U_s(X_k)| \leq s - r$ by Remark 3.1(d). Hence by (4.1) and a trivial calculation, $|m_r - m_s| \leq s - r$. Hence by (4.2) and a simple calculation, for any $k \in \mathbb{Z}$, $|Y_{k,r} - Y_{k,s}| \leq 2(s - r)$. Hence for any $n \in \mathbb{N}$, $|(\sum_{k=1}^{n} Y_{k,r}) - (\sum_{k=1}^{n} Y_{k,s})| \leq 2n(s-r)$ and hence $\|(\sum_{k=1}^{n} Y_{k,r}) - (\sum_{k=1}^{n} Y_{k,s})\|_2 \leq 2n(s-r)$. Hence for any $n \in \mathbb{N}$,

$$\Big|\|\sum_{k=1}^{n} Y_{k,r}\|_2 - \|\sum_{k=1}^{n} Y_{k,s}\|_2\Big| \leq 2n(s-r). \tag{4.13}$$

That was shown for arbitrary $0 \leq r < s$. It follows that for any given $n \in \mathbb{N}$, the mapping $r \mapsto \|\sum_{k=1}^{n} Y_{k,r}\|_2$ for $r \in [0, \infty)$ is (uniformly) continuous.

Now by Remark 3.2(c), $E[|U_r(X_0)|^2] \to 0$ as $r \to \infty$. Hence by the "second half" of (4.3), $\|Y_{0,r}\|_2 \to 0$ as $r \to \infty$. Hence for any given positive integer n, $\|\sum_{k=1}^{n} Y_{k,r}\|_2 \leq n \cdot \|Y_{0,r}\|_2 \to 0$ as $r \to \infty$. Also, by (4.8) and (4.9), for any $n \geq N_0$, $E[(\sum_{k=1}^{n} Y_{k,0})^2] > 1$ and hence $\|\sum_{k=1}^{n} Y_{k,0}\|_2 > 1$. Hence by the Intermediate Value Theorem and the second sentence after (4.13), for each integer $n \geq N_0$, there exists a positive number $r(n)$ such that (4.10) holds. The completes the proof of the first sentence of Lemma 4.3.

Now to prove the rest of Lemma 4.3, suppose $(r(n), n \geq N_0)$ is a sequence of positive numbers such that (4.10) holds for all $n \geq N_0$. By (4.8) and (4.10), one has that for any $n \geq N_0$, $(1/C)nE[Y_{0,r(n)}^2] \leq 1 \leq CnE[Y_{0,r(n)}^2]$. Eq. (4.12) follows trivially. All that remains is to prove (4.11).

Suppose instead that (4.11) fails to hold; we shall aim for a contradiction. There exists an infinite set $\Lambda \subset \{N_0, N_0+1, N_0+2, \dots\}$ and a positive number A such that $r(n) \leq A$ for all $n \in \Lambda$. By stationarity and the second sentence after (4.13), applied with $n = 1$, the mapping $r \mapsto \|Y_{0,r}\|_2$ for $r \in [0, \infty)$ is continuous. Hence by the inequality in (4.3) and the compactness of the interval $[0, A]$ there exists a positive number B such that for all $r \in [0, A]$, $\|Y_{0,r}\|_2 \geq B$. In particular, $\|Y_{0,r(n)}\|_2 \geq B$ for all $n \in \Lambda$. Hence by (4.8), $E[(\sum_{k=1}^{n} Y_{k,r(n)})^2] \geq B^2 n/C$ for all $n \in \Lambda$. But (since Λ is unbounded above) that contradicts (4.10). Hence (4.11) must hold after all. That completes the proof of Lemma 4.3.

Step 4.4. Conclusion of proof of Theorem 2.1. Applying Lemma 4.3, let $(r(n), n \geq N_0)$ be a sequence of positive numbers that satisfies (4.10). By Lemma 4.3, this sequence satisfies (4.11) and (4.12). By (4.11), (4.12) and (4.5), $\limsup_{n\to\infty} n \cdot G(r(n)) < \infty$. Hence by (4.6) and stationarity, the Lindeberg condition holds:

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \sum_{k=1}^{n} E[Y_{k,r(n)}^2 I(|Y_{k,r(n)}| \geq \varepsilon)] = 0. \tag{4.14}$$

Also, by (4.10) and (4.12) and stationarity (recall the inequality in (4.3)),

$$\liminf_{n\to\infty} \frac{E\Big[\Big(\sum_{k=1}^{n} Y_{k,r(n)}\Big)^2\Big]}{\sum_{k=1}^{n} E[(Y_{k,r(n)})^2]} > 0. \tag{4.15}$$

By (4.10), (4.14), (4.15), and Theorem 6.4 in the Appendix (Section 6), regardless of which of the two dependence assumptions (i), (ii) in Theorem 2.1 is assumed, eq. (4.4) holds. That completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

As in the hypothesis of Theorem 2.2, suppose $0 < \lambda < 1$. Define the number $\theta \in (0, 1/4)$ by

$$\theta := \lambda/4. \tag{5.1}$$

Let $\mathbf{V} := (V_k, k \in \mathbf{Z})$ be a strictly stationary Markov chain with state space $\{1, 2, 3\}$, with marginal distribution given by

$$P(V_0 = 1) = \frac{1}{1 + 2\theta} \quad \text{and} \quad P(V_0 = 2) = P(V_0 = 3) = \frac{\theta}{1 + 2\theta}, \tag{5.2}$$

and with one-step transition probabilities $p_{ij} := P(V_1 = j | V_0 = i)$, for $i, j \in \{1, 2, 3\}$, given by

$$p_{11} = 1 - \theta, \quad p_{12} = \theta, \quad \text{and} \quad p_{23} = p_{31} = 1. \tag{5.3}$$

It is easy to check that the marginal distribution in (5.2) is the invariant distribution for the transition probabilities given in (5.3). It is easy to see that this Markov chain $\mathbf{V}$ is irreducible and aperiodic.

To avoid frivolous technicalities, we shall henceforth assume that for *every* $\omega \in \Omega$ and $k \in \mathbb{Z}$, the ordered pair $(V_k(\omega), V_{k+1}(\omega))$ is either $(1,1)$, $(1,2)$, $(2,3)$, or $(3,1)$.

Let $\mathbf{Z} := (Z_k, k \in \mathbb{Z})$ be a sequence of independent $N(0,1)$ random variables, with this sequence $\mathbf{Z}$ being independent of the Markov chain $\mathbf{V}$.

Define the sequence $\mathbf{X} := (X_k, k \in \mathbb{Z})$ as follows: For each $k \in \mathbb{Z}$,

$$X_k := \begin{cases} 0 & \text{if } V_k = 1 \\ Z_k & \text{if } V_k = 2 \\ -Z_{k-1} & \text{if } V_k = 3. \end{cases} \tag{5.4}$$

By an elementary argument, this sequence $\mathbf{X}$ is strictly stationary.

For events A and B, the notation $A \doteq B$ will mean that $P(A \triangle B) = 0$, where $\triangle$ denotes symmetric difference. From (5.4) and (5.1), one has that for any given integer k,

$$\{X_k = 0\} \doteq \{V_k = 1\}. \tag{5.5}$$

Now X_0 is square-integrable and unbounded, and hence (2.10) holds by (2.9) and (3.1).

The proof that this sequence $\mathbf{X}$ satisfies (2.11) is simply (with one "trivial adjustment") the observation in Housworth and Shao [2000, first sentence after Theorem 1]. For the $N(0,1)$ density function $\phi(x)$, one has that for every $\varepsilon > 0$, $\lim_{x\to\infty} \phi(x+\varepsilon)/\phi(x) = 0$. Now by (5.4) and (5.2), the distribution of X_0 is a mixture (convex combination) of the $N(0,1)$ distribution and the point mass at 0; and hence it follows by symmetry and a simple calculation that for every $\varepsilon > 0$, $\lim_{x\to\infty} P(|X_0| > x+\varepsilon)/P(|X_0| > x) = 0$. It follows from a further simple integration that (2.11) holds.

Now what remains is to prove properties (a), (b), and (c) in the statement of Theorem 2.2. We shall start with property (b).

Proof of property (b). Since $\rho^*(\mathbf{Z}, 1) = 0$ (because of the independence of the Z_k's), one has that for each $n \ge 2$, by (5.4) and Bradley [2007, v1, Theorem 6.1],

$$\rho^*(\mathbf{X}, n) \le \max\{\rho^*(\mathbf{V}, n),\ \rho^*(\mathbf{Z}, n-1)\} = \rho^*(\mathbf{V}, n).$$

Property (b) now follows from Bradley [2007, v1, Theorem 7.15] (applied to the Markov chain $\mathbf{V}$). What remains is to prove properties (a) and (c).

Proof of property (a). Refer again to (2.1) and (2.4). Let the events $A \in \sigma(X_k, k \le 0)$ and $B \in \sigma(X_k, k \ge 1)$ be arbitrary but fixed. To complete the proof of property (a), it suffices to prove that $|P(A \cap B) - P(A)P(B)| \le \lambda$. By (5.1) and the triangle inequality, it suffices to prove

$$|P(A \cap \{X_0 = 0\} \cap B) - P(A \cap \{X_0 = 0\})P(B)| \le 2\theta \tag{5.6}$$

and

$$|P(A \cap \{X_0 \ne 0\} \cap B) - P(A \cap \{X_0 \ne 0\})P(B)| \le 2\theta. \tag{5.7}$$

Between the absolute value signs in the left side of (5.7), is the difference of two nonnegative terms that are each trivially bounded above by $P(X_0 \ne 0)$. Hence the left side of (5.7) is bounded above by $P(X_0 \ne 0)$. By (5.5) and (5.2), $P(X_0 \ne 0) = 2\theta/(1 + 2\theta) < 2\theta$. Thus (5.7) holds. To complete the proof of (a), what remains is to prove (5.6).

In the proof of (5.6), for σ-fields $\mathcal{A}$ and $\mathcal{B}$, the notation $\mathcal{A} \vee \mathcal{B}$ will mean the σ-field generated by $\mathcal{A} \cup \mathcal{B}$.

Now (see e.g. Bradley [2007, v1, Appendix, Remark A042]) by (5.5) and (5.2) the event $\{X_0 = 0\}$ is an atom of the sigma field $\sigma(X_0)$. Hence (see e.g. Bradley [2007, v1, Appendix, Theorem A036]) there exists an event $A^* \in \sigma(X_k, k \le -1)$ (henceforth fixed) such that $A \cap \{X_0 = 0\} \doteq A^* \cap \{X_0 = 0\}$. By (5.5) and (5.4), one now has that $A \cap \{X_0 = 0\} \doteq A^* \cap \{V_0 = 1\}$, and also this latter event satisfies $A^* \cap \{V_0 = 1\} \in \sigma(V_k, k \le 0) \vee \sigma(Z_k, k \le -1)$. Also by (5.4), $B \in \sigma(V_k, k \ge 1) \vee \sigma(Z_k, k \ge 0)$. Hence by Bradley [2007, v1, Theorem 6.2(a)],

$$\begin{aligned}
& [\text{LHS of } (5.6)] \\
= \;& |P(A^* \cap \{V_0 = 0\} \cap B) - P(A^* \cap \{V_0 = 0\})P(B)| \\
\le \;& \alpha\Big(\sigma(V_k, k \le 0) \vee \sigma(Z_k, k \le -1), \sigma(V_k, k \ge 1) \vee \sigma(Z_k \ge 0)\Big) \\
\le \;& \alpha(\mathbf{V}, 1) + \alpha(\mathbf{Z}, 1) \;=\; \alpha(\mathbf{V}, 1) + 0 \;=\; \alpha(\mathbf{V}, 1)
\end{aligned} \tag{5.8}$$

(and hence as a simple consequence the second inequality in (5.8) is really an equality).

By a well known property of (strictly stationary) Markov chains (see e.g. Bradley [2007, v1, Theorem 7.3(a)]), $\alpha(\mathbf{V}, 1) = \alpha(\sigma(V_0), \sigma(V_1))$. The event $\{V_0 = 1\}$ is an atom of the σ-field $\sigma(V_0)$, and by (5.2) it satisfies $P(V_0 = 1) \ge 1 - 2\theta$. Hence (see e.g. Bradley [2007, v1, Proposition 3.19(a)]),

$\alpha(\sigma(V_0), \sigma(V_1)) \le 2\theta$. Eq. (5.6) now follows from (5.8). That completes the proof of property (a).

Proof of property (c). This is the final task in the proof of Theorem 2.2.

Suppose $n \in \mathbb{N}$ and $r \ge 0$.

Suppose $\omega \in \Omega$ is such that $V_1(\omega) = V_n(\omega) = 1$. Then the vector $(V_1(\omega), V_2(\omega), \dots, V_n(\omega))$ is either equal to $(1, 1, \dots, 1)$ or consists of 1's occasionally interrupted by occurrences of the ordered pair $(2, 3)$. If $k \in \{1, 2, \dots, n\}$ is such that $V_k(\omega) = 1$, then $U_r(X_k(\omega)) = 0$ by (5.4). If ($n \ge 2$ and) $k \in \{1, 2, \dots, n-1\}$ is such that $(V_k(\omega), V_{k+1}(\omega)) = (2, 3)$, then by (5.4) and Remark 3.1(b),

$$\begin{aligned} U_r(X_k(\omega)) + U_r(X_{k+1}(\omega)) &= U_r(Z_k(\omega)) + U_r(-Z_k(\omega)) \\ &= U_r(Z_k(\omega)) - U_r(Z_k(\omega)) \;=\; 0. \end{aligned}$$

Hence $\sum_{k=1}^{n} U_r(X_k(\omega)) = 0$.

What has been shown here is that $\{V_1 = V_n = 1\} \subset \{\sum_{k=1}^{n} U_r(X_k) = 0\}$. Now by (5.2),

$$P(V_0 = V_n = 1) = 1 - P(\{V_0 \in \{2, 3\}\} \cup \{V_n \in \{2, 3\}\}) \ge 1 - 4\theta.$$

Property (c) now follows from (5.1). That completes the proof of Theorem 2.2.

Remark 5.1. For the (strictly stationary) sequence $\mathbf{X}$ constructed above, the equality $\rho(\mathbf{X}, 1) = \rho^*(\mathbf{X}, 1) = 1$, pointed out in Section 2 in connection with Theorem 2.2, has the following brief explanation: By (5.2), (5.3), (5.4), and (5.5), the (nondegenerate) random variables (indicator functions) $I(\{X_{-1} = 0\} \cap \{X_0 \ne 0\})$ and $I(\{X_1 \ne 0\} \cap \{X_2 = 0\})$ are equal a.s., and hence their correlation equals 1.

6. Appendix

This section just gives convenient statements of certain results in the literature on mixing that are used in the proof of Theorem 2.1.

Lemma 6.1. *Suppose* $(q(1), q(2), q(3), \dots)$ *is a non-increasing sequence of nonnegative numbers such that* $q(1) < 1$ *and* $\sum_{n=1}^{\infty} q(2^n) < \infty$. *Then there exists a positive constant* $C = C(q(1), q(2), q(3), \dots)$ *such that the following holds:*

If $\mathbf{X} := (X_k, k \in \mathbb{Z})$ *is a strictly stationary sequence of centered, square-integrable random variables such that* $\rho(\mathbf{X}, n) \le q(n)$ *for all* $n \in \mathbb{N}$, *then for* *every $n \in \mathbb{N}$, one has that*

$$(1/C)nE[X_0^2] \le E\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] \le CnE[X_0^2].$$

One reference for Lemma 6.1 is Bradley [1988, Lemmas 2.1 and 2.3]. The upper bound (which does not require the assumption $q(1) < 1$) was known earlier, e.g. from the work of Ibragimov [1975]; and in Peligrad [1982, Lemma 3.4] it was shown (with suitable reformulation) in a more general form, not requiring stationarity.

Lemma 6.2. *Suppose $\mathbf{X} := (X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of centered, square-integrable random variables such that $\rho^*(1) < 1$. Then for every positive integer n,*

$$\frac{[1-\rho^*(1)]}{[1+\rho^*(1)]}\, E[X_0^2] \le E\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] \le \frac{[1+\rho^*(1)]}{[1-\rho^*(1)]}\, E[X_0^2].$$

Lemma 4.2 can be found in Bradley [1992, Lemma 2.1] or Bradley [2007, v1, Lemma 8.21]. In fact it follows implicitly from earlier results of Moore [1963] involving spectral density.

Context 6.3. Suppose $\xi := (\xi_{n,i}, n \in \mathbb{N}, i \in \{1, 2, \dots, k_n\})$ is a triangular array of centered, square-integrable random variables, where $(k_1, k_2, k_3, \dots)$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. For each positive integer n, define the nonnegative number σ_n by

$$\sigma_n^2 := E\Big[\Big(\sum_{i=1}^{k_n} \xi_{n,i}\Big)^2\Big].$$

For each positive integer m, define the following dependence coefficients:

$$\begin{aligned}
\alpha(\xi, m) &:= \sup_{\{n:k_n \ge m+1\}} \sup_{1 \le u \le k_n - m} \alpha\Big(\sigma(\xi_{n,i}, 1 \le i \le u), \\
&\qquad\qquad \sigma(\xi_{n,i}, u+m \le i \le k_n)\Big); \\
\rho(\xi, m) &:= \sup_{\{n:k_n \ge m+1\}} \sup_{1 \le u \le k_n - m} \rho\Big(\sigma(\xi_{n,i}, 1 \le i \le u), \\
&\qquad\qquad \sigma(\xi_{n,i}, u+m \le i \le k_n)\Big); \\
\rho^*(\xi, m) &:= \sup_{\{n:k_n \ge m+1\}} \sup \rho\Big(\sigma(\xi_{n,i}, i \in S),\, \sigma(\xi_{n,i}, i \in T)\Big)
\end{aligned}$$

where in the last line the inner supremum is taken over all pairs of nonempty, disjoint sets $S, T \subset \{1, 2, \ldots, k_n\}$ such that $\mathrm{dist}(S, T) := \min_{s \in S, t \in T} |s - t| \geq m$.

Theorem 6.4. *Assume that all assumptions in the first sentence of Context 6.3 hold. Then in the notations of Context 6.3, the following statement holds:*

Suppose that $\sum_{i=1}^{k_n} E[\xi_{n,i}^2] > 0$ for all n sufficiently large, and that

$$\liminf_{n \to \infty} \frac{\sigma_n^2}{\sum_{i=1}^{k_n} E[\xi_{n,i}^2]} > 0. \tag{6.1}$$

Suppose that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} \sum_{i=1}^{k_n} E[\xi_{n,i}^2 I(|\xi_{n,i}| \geq \varepsilon)] = 0 \tag{6.2}$$

(where $I(\ldots)$ denotes the indictor function).

Suppose also that (at least) one of the following two conditions (i), (ii) holds:

(i) $\sum_{m=1}^{\infty} \rho(\xi, 2^m) < \infty$; or

(ii) $\rho^*(\xi, m) < 1$ for some $m \geq 1$, and $\alpha(\xi, m) \to 0$ as $m \to \infty$.

Then $(1/\sigma_n) \sum_{i=1}^{k_n} \xi_{n,i} \Rightarrow N(0, 1)$ as $n \to \infty$.

Eq. (6.2) is the Lindeberg condition for this context. Under the mixing hypothesis (i), Theorem 6.4 is due to Utev [1990, Theorem 4.1]. Under the mixing hypothesis (ii), Theorem 6.4 is due to Peligrad [1996, Theorem 2.1] (see also Theorem 2.2 in that paper).

When his paper Utev [1990] was published, Utev himself pointed out (in a private communication to one of the authors, R.C.B.) that eq. (4.3) in his statement of his Theorem 4.1 was intended to be (6.1) above, but had a serious typographical error; and he called attention to eq. (4.6) in his proof of his Theorem 4.1 to show where the intended correct version of his eq. (4.3) (namely (6.1) above) was used.

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